B. Sc. Part III (Hons) 7th Paper

DIFF. EQNS.

Q. Solve and find the complete integral of the following
$$p^2+q^2-2px-2qy+2xy=0.$$

Soln.

Let $f \equiv p^2+q^2-2px-2qy+2xy=0$ ——— (1)

Charpit's auxiliary equations are

$$\frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}}=\frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}}=\frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}}=\frac{dx}{-\frac{\partial f}{\partial p}}=\frac{dy}{-\frac{\partial f}{\partial q}}$$

$$\Rightarrow \frac{dp}{-2p+2y+p.0}=\frac{dq}{-2q+2x+q.0}=\frac{dz}{-p(2p-2x)-q(2q-2y)}=\frac{dx}{-(2p-2x)}=\frac{dy}{-(2q-2y)}$$

$$\Rightarrow \frac{dp}{-p+y}=\frac{dq}{x-q}=\frac{dz}{-p(p-x)-q(q-y)}=\frac{dx}{x-p}=\frac{dy}{y-q}$$

$$\Rightarrow \frac{dp+dq}{x+y-(p+q)}=\frac{dx+dy}{(x+y)-(p+q)} \Rightarrow dp+dq=dx+dy$$

Integrating, we get

$$p+q=x+y+a \quad \text{——— (2)}$$

From (1), $(p-x)^2+(q-y)^2=(x-y)^2$

$$\Rightarrow (p-x)^2+[a-(p-x)]^2=(x-y)^2$$

$$\Rightarrow 2(p-x)^2-2a(p-x)+[a^2-(x-y)^2]=0$$

which is a quadratic eqn in $(p-x)$.

$$\therefore p-x = \frac{2a \pm \sqrt{4a^2 - 4.2.\{a^2-(x-y)^2\}}}{4}$$

$$\Rightarrow p-x = \frac{1}{2}\left[a \pm \sqrt{2(x-y)^2 - a^2}\right.$$

$$\left[\because \sqrt{4a^2 - 8\{a^2-(x-y)^2\}} = 2\left[\sqrt{a^2 - 2a^2 + 2(x-y)^2}\right]\right.$$
$$\left. = 2\sqrt{2(x-y)^2 - a^2}\right]$$

Taking $(+)$ve sign only, we have

$$p = x + \frac{1}{2}\left[a + \sqrt{2(x-y)^2 - a^2}\right] \quad \text{—— (3)}$$

Putting this value in (2), we have

$$q = y + \frac{1}{2}\left[a - \sqrt{2(x-y)^2 - a^2}\right] \quad \text{—— (4)}$$

$$\because \quad dz = p\,dx + q\,dy$$

$$\Rightarrow dz = x\,dx + y\,dy + \frac{a}{2}(dx+dy) + \frac{1}{2}\left[\sqrt{2(x-y)^2 - a^2}\right](dx-dy)$$

$$\Rightarrow z = \frac{x^2}{2} + \frac{y^2}{2} + \frac{a}{2}(x+y) - \frac{a^2}{4}\log\left\{(x-y) + \sqrt{(x-y)^2 - \frac{a^2}{2}}\right.$$
$$+ K_1$$

where K_1 is constant of integration.

This is the complete integral of the given diff. eqn.